→ Elementary Matrices (3 types)

elementary row operation.

1st Type Elementary Matrices: $I_n \xrightarrow{\text{apply (1)}} E$ — 1st type elem. row operation $(r_i \leftrightarrow r_j)$

2nd Type Elementary Matrices: $I_n \longrightarrow E$ — single 2nd type row op. $(cr_i \to r_i)$

3rd Type Elementary Matrices: $I_n \longrightarrow E$ — single 3rd type row op. $(cr_i + r_j \to r_j)$

Ex

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}}_{I_4} \xrightarrow{r_2 \leftrightarrow r_4} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{E} \to$$ An elementary matrix of 1st type

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}}_{I_3} \xrightarrow{1/2\, r_2 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$$ an elementary matrix of 2nd type

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}}_{I_4} \xrightarrow{-3r_1 + r_4 \to r_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \to$$ an elementary matrix of 3rd type

! Every elementary matrix corresponds to the row operation that creates it.

any matrix

$$A \xrightarrow{\text{a row operation}} A' \quad \Leftrightarrow \quad EA = A'$$

left multiplication

Ex

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -1 & 2 & 4 \\ 3 & 4 & 0 & -2 \end{bmatrix}_{3\times4} \xrightarrow{-3r_1 + r_3 \to r_3} \begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -1 & 2 & 4 \\ 0 & 10 & -9 & -2 \end{bmatrix} \to ?$$

row op.

$I_n \to$ square

$E_{3\times3} A_{3\times4}$

$E_{2\times2}$ $(2)\times4$

$$I_3 \xrightarrow{-3r_1 + r_3 \to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = E_1 \to$$ elementary matrix

$E_1 A = ?$

$= \checkmark$

$I_3 \longrightarrow \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = E_1$ elementary matrix $= \checkmark$

$E_1 A = ?$

$$E_1 A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -1 & 2 & 4 \\ 3 & 4 & 0 & -2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -1 & 2 & 4 \\ 0 & 10 & -9 & -2 \end{bmatrix}}_{E_1 A}$$

3x3 3x4 3 x 4

$-3+0+3 \quad 6+0+4 \quad -9+0+0 \quad 0+0+-2$

$$\underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -1 & 2 & 4 \\ 0 & 10 & -9 & -2 \end{bmatrix}}_{E_1 A} \xrightarrow{-5r_1 + r_2 \to r_2} \underbrace{\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 9 & -13 & 4 \\ 0 & 10 & -9 & -2 \end{bmatrix}}_{E_2(E_1 A)} \dashrightarrow$$

3x4

$$I_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

Finding A^{-1} Using Elementary Matrices

! $A_{n \times n} \xrightarrow{REF} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \ddots \\ & & & & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots 1 \end{bmatrix} = I_n$ → this is one possibility

$RREF(A) = I_n$

Don't forget that this is not the only possibility

$A_{n \times n} \to \begin{bmatrix} 1 & \cancel{0} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix} \neq I_n$ → RREF does not reach to I_n.

→ If you are able to obtain I_n as the RREF of A;

$$A_{n \times n} \xrightarrow{E_1} \underbrace{[\quad]}_{E_1 A} \xrightarrow{E_2} \underbrace{[\quad]}_{E_2(E_1 A)} \xrightarrow{E_3} \underbrace{[\quad]}_{E_3(E_2(E_1 A))} \dashrightarrow \xrightarrow{E_k} \underbrace{\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots 1 \end{bmatrix}}_{RREF(A)} = I_n$$

k steps

$$\underbrace{E_k \cdots E_3 E_2 E_1}_{?} A = I_n$$

A^{-1}

$$E_k \cdots E_3 E_2 E_1 = A^{-1}$$

$$E_k \cdots E_3 E_2 E_1 \underset{\text{identity}}{I_n} = A^{-1}$$

$(I_n \xrightarrow{E_1} \xrightarrow{E_2} \longrightarrow \xrightarrow{E_k} A^{-1})$

! $[A \mid I_n] \xrightarrow{E_1} \quad \longrightarrow [RREF(A) = I_n \mid A^{-1}]$

!

$$[A \mid I_n] \xrightarrow{E_1} \; \xrightarrow{E_2} \cdots \longrightarrow [\underbrace{RREF(A)}_{=I_n} \mid A^{-1}]$$

$$A \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 2 & 0 & 1 \end{bmatrix}_{3\times3} \longrightarrow \longrightarrow \quad RREF(A) = I_n$$

at most ~~at 7 steps~~ you will be able to reach A^{-1}

easier than solving a 9×9 system of LE.

! $A_{n\times n}$ If $RREF(A) = I_n \iff A^{-1}$ exists. → A is invertible $[A \mid I_n] \to \to [I_n \mid A^{-1}]$

If $RREF(A) \neq I_n \iff A^{-1}$ does not exist. → A is singular

Ex $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 2 & 4 \end{bmatrix}_{3\times3}$ Is A invertible? If so, find A^{-1}.

Use elementary matrices $[A \mid I] \to$

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array}\left[\begin{array}{ccc|ccc} 2 & 1 & -2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 1 & 0 \\ -2 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

A $\quad I_3$

we will try to get RREF of A

$$\xrightarrow[2r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -5 & -2 & 1 & -2 & 0 \\ 0 & 8 & 4 & 0 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 & 2/5 & 0 \\ 0 & 8 & 4 & 0 & 2 & 1 \end{array}\right]$$

$$AA^{-1} = I_n$$

$$\xrightarrow{-8r_2 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 4/5 & 8/5 & -6/5 & 1 \end{array}\right] \xrightarrow{\frac{5}{4}r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 2 & -3/2 & 5/4 \end{array}\right]$$

REF

$$\xrightarrow{-\frac{2}{5}r_3 + r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1/2 \\ 0 & 0 & 1 & 2 & -3/2 & 5/4 \end{array}\right] \xrightarrow{-3r_2 + r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 3/2 \\ 0 & 1 & 0 & -1 & 1 & -1/2 \\ 0 & 0 & 1 & 2 & -3/2 & 5/4 \end{array}\right]$$

$=I_3 \qquad A^{-1}$

$$A^{-1} = \begin{bmatrix} 3 & -2 & 3/2 \\ -1 & 1 & -1/2 \end{bmatrix} ✓$$

check answer $AA^{-1} \stackrel{?}{=} I_n$

$$A^{-1} = \begin{bmatrix} 3 & -2 & 3/2 \\ -1 & 1 & -1/2 \\ 2 & -3/2 & 5/4 \end{bmatrix} \checkmark$$

check your answer $\quad AA^{-1} \stackrel{?}{=} I_n$

to check

$$AA^{-1} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 2 & 4 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 3 & -2 & 3/2 \\ -1 & 1 & -1/2 \\ 2 & -3/2 & 5/4 \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2+3+0$ $\quad 6-1-4 \quad -4+1+3 \quad 3 - 1/2 - 5/2$

$3 - \frac{3}{2} - \frac{3}{2} + 0$

$-6-2+8 \quad 4+2-6 \quad -3-1+5$

Ex

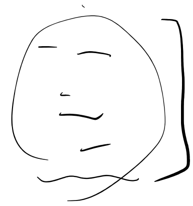
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}_{4\times 4}$$

Is A invertible? If yes, find A^{-1}.

$$\underbrace{\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right.}_{A} \underbrace{\left|\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{I} \xrightarrow[-3r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & -8 & -12 & -3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

RREF

$$\xrightarrow{r_2 \leftrightarrow r_4} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \\ 0 & -1\;0 & -8\;(-6) & -12 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{r_2+r_3 \to r_3} \dashrightarrow \left[\begin{array}{cccc} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \\ 0 & 0 & 0 & 0 \end{array}\right]$$



$\mathrm{RREF}(A) \neq I_4$

Since $\mathrm{RREF}(A) \neq I_4$, A^{-1} does not exist.

Some conclusions about System of Linear Equations

matrices $\rightleftarrows$ Systems of LE

m eqns, n unknowns

m×n system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\quad + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots\quad + a_{mn}x_n &= b_m \end{aligned}$$

$\updownarrow$

represents our system of the eqns $\rightarrow$ $\boxed{A_{m\times n}\, x_{n\times 1} = b_{m\times 1}}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

A = coefficient matrix

X = column matrix of variables

b = column matrix of results

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n =$

represents our system of lin. eqns. $\rightarrow$ $\boxed{A_{m\times n}\, x_{n\times 1} = b_{m\times 1}}$

variables

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n =$

non-homogeneous

$\boxed{Ax = b}$

1) unique solution
2) inf. many solutions
3) no solution

Homogeneous System

$\boxed{Ax = 0}$

1) unique solution = trivial soln.
2) inf. many solutions (includes the trivial soln)

~~3~~

if $b_1 = b_2 = \dots = b_m = 0$

$x_1 = ?$
$x_2 = ?$
$x_n = ?$

$a_{11}x_1 + \dots = 0$ ✓
$a_{12}x_2 + \dots = 0$ ✓
$= 0$ ✓
$= 0$ ✓

$x_1 = x_2 = \dots = x_n = 0 \Rightarrow$ trivial soln.

If the system has a square coefficient matrix ($A_{n\times n}$)

$A_{n\times n}\, x = b$

if A^{-1} exists; multiply both sides from the left with A^{-1};

$\underbrace{A^{-1}A}_{I_n} x = A^{-1}b \Rightarrow \boxed{x = A^{-1}b} \rightarrow$ unique soln.

$A_{n\times n} \rightarrow$ coef. matrix $\quad Ax = b$ or $Ax = 0$

If $RREF(A) = I_n \Leftrightarrow A^{-1}$ exists $\Leftrightarrow Ax = b$ has a unique soln. $= A^{-1}b$
$\Leftrightarrow Ax = 0$ " " " " = the trivial solution $\Rightarrow 0$

If $RREF(A) \neq I_n \Leftrightarrow A^{-1}$ does not exist $\Leftrightarrow Ax = b$ has either inf. many solutions or No solution
$\Leftrightarrow Ax = 0$ has infinitely many solutions.

1. Which of the matrices that follow are elementary matrices? Classify each elementary matrix by type.

(a) $I_2 \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$ Elem. mat. of 1st type

(b) $I_2 \rightarrow$ ~~$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$~~

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ only 1 row op. NO!

~~$2r_1 \rightarrow r_1$, $3r_2 \rightarrow r_2$~~

(c) $I_3 \xrightarrow[5r_1 + r_3 \rightarrow r_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$ Elementary matrix of 3rd type

(d) $I_3 \xrightarrow{5r_2 \rightarrow r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ elem. matrix of 2nd type

3. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$.

$E = ?$
$EA = B$
$A \longrightarrow B$

$I_2 \xrightarrow{-2r_1 \rightarrow r_1} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = E$ ✓

(a) $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

elementary matrix E such that $EA = B$.

(a) $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

$A \xrightarrow{-2r_1 \to r_1} B$

$EA = B$

$A \xrightarrow[\text{row op.}]{} B$

$I_2 \longrightarrow \begin{bmatrix} \cdot & \cdot \\ 0 & 1 \end{bmatrix} = E$ ✓

4th Week Friday

17 Mart 2023 Cuma 10:50

$[A \mid I] \longrightarrow [\underset{=I_n}{RREF} \mid A^{-1}]$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix} \qquad A^{-1}$$

$$[A \mid I] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2r_1+r_3 \to r_3]{-3r_1+r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & -3 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & -1 & 1/3 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-2r_2+r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & -1 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & -2/3 & 1 \end{array}\right]$$

$$\xrightarrow{3r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/3 & -1 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right] \xrightarrow[-1/3 r_3 + r_2 \to r_2]{-r_3+r_1 \to r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & -2 & 3 \end{array}\right]$$

RREF $= I_3$ $\qquad$ A^{-1}

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark \quad \text{to verify}$$

$6+3-8$

$Ax = b$

(b) Use A^{-1} to solve $Ax = \mathbf{b}$ for the following choices of $\mathbf{b}$.

(i) $\mathbf{b} = (1, 1, 1)^T$ (ii) $\mathbf{b} = (1, 2, 3)^T$

(iii) $\mathbf{b} = (-2, 1, 0)^T$

$x = A^{-1}b$

$\to$ (i) $\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$x = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

unique sol.

$x_1 = 0 \quad x_2 = -1 \quad x_3 = 1$

$Ax = b$

$\underbrace{A^{-1}A}_{I_n} x = A^{-1}b$

$\boxed{x = A^{-1}b}$

Inverses of Elementary Matrices

$[E \mid I] \longrightarrow [I \mid E^{-1}]$

1st type

Ex

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$I \qquad E^{-1}$

$\boxed{E = E^{-1}}$

$$EE^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

! For elementary matrices of the 1st type $\boxed{E^{-1} = E}$

! For elementary matrices of the 1 type $\boxed{E^{-1} = E}$

2nd Type

Ex

$$I_4 \xrightarrow{4r_3 \to r_3} E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{4}r_3 \to r_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

E^{-1}

! For elementary matrices of the 2nd type $E(cr_i \to r_i) \to E^{-1}(1/c\, r_i \to r_i)$

3rd Type

Ex

$$I_3 \xrightarrow{-3r_1 + r_3 \to r_3} E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3r_1 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$$

E^{-1}

! For elementary matrices of the 3rd type $E(cr_i + r_j \to r_j) \to E^{-1}(-cr_i + r_j \to r_j)$

LU-factorization

$$A_{n\times n} = LU$$

lower triangular

upper triangular

$A = LU$

$A_{n\times n} \longrightarrow$ row operation only of 3rd type $\qquad E_1 \quad E_2 \quad E_3$

$$\begin{bmatrix} \# & & \\ 0 & \# & \\ 0 & 0 & \# \end{bmatrix} = \mathcal{U}$$

strict triangular

make zeroes only (Not leading 1's)

$$E_3E_2E_1A = \mathcal{U}$$

$$\underbrace{E_3^{-1}E_3}_{I} E_2 E_1 A = E_3^{-1}\mathcal{U}$$

$$\underbrace{E_2^{-1}E_2}_{I} E_1 A = E_2^{-1}E_3^{-1}\mathcal{U}$$

$$\underbrace{E_1^{-1}E_1}_{I} A = E_1^{-1}E_2^{-1}E_3^{-1}\mathcal{U}$$

$$A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}}_{L}\mathcal{U}$$

Ex

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow[-2r_1 + r_3 \to r_3 \to E_2]{-3r_1 + r_2 \to r_2 \to E_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[E_3]{r_2 + r_3 \to r_3} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \mathcal{U}$$

upper triangular

3rd type

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$L = E_1^{-1}E_2^{-1}E_3^{-1}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$$